The extension of traces for Sobolev mappings between manifolds

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Given compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ and $p \in (1, \infty)$, the question of traces for Sobolev mappings consists in characterising the mappings from $\partial\mathcal{M}$ to $\mathcal{N}$ that can arises of maps in the first-order Sobolev space $\dot{W}^{1,p}(\mathcal{M}, \mathcal{N})$. A direct application of Gagliardo's characterisation of traces for the linear spaces $\dot{W}^{1,p}(\mathcal{M}, \mathbb{R})$ shows that traces of maps in $\dot{W}^{1,p}(\mathcal{M}, \mathcal{N})$ should belong to the fractional Sobolev-Slobodeckiĭ space $\dot{W}^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$. There is however no reason for Gagliardo's linear extension to satisfy the nonlinear constraint imposed by $\mathcal{N}$ on the target.

In the case $p > \dim \mathcal{M}$, Sobolev mappings are continuous and thus traces of Sobolev maps are the mappings of $\dot{W}^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ that are also restrictions of continuous functions [2]. The critical case $p = \dim \mathcal{M}$ can be treated similarly thanks to their vanishing mean oscillation property [2, 3, 6].

The case $1 < p < \dim \mathcal{M}$ is more delicate. It was first proved that when the first homotopy $\pi_1(\mathcal{N}), \dots, \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ are *trivial,* then the trace operator from $\dot{W}^{1,p}(\mathcal{M}, \mathcal{N})$ to $\dot{W}^{1-1/p,p}(\partial\mathcal{M}, \mathcal{N})$ is surjective [4]. On the other hand, several conditions for the surjectivity have been known: topological obstructions require $\pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ to be *trivial* [2, 4] whereas analytical obstructions arise unless the groups $\pi_1(\mathcal{N}), \dots, \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$ are *finite* [1] and, when $p \geq 2$ is an integer, $\pi_{p-1}(\mathcal{N})$ is *trivial* [5].

In a recent work, I have completed the characterisation of the cases where the trace is surjective, proving that the known necessary conditions turn out to be sufficient [7]. I extend the traces thanks to a new construction which works on the domain rather than in the image. When $p \geq \dim \mathcal{M}$ the same construction also provides a Sobolev extension with linear estimates for maps that have a continuous extension, provided that there are no known analytical obstructions to such a control.

References

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