Persistent homology of simplicial maps with Dowker Complexes

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Consider the sequence of simplicial maps :

$$K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} K_3 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} K_n. \quad (1)$$

Our goal is to compute the persistent homology of (1). If all f_i are inclusion maps, then (1) is a filtration. In this cases, there is an efficient algorithm to compute the persistence. In [1], the authors decomposed each simplicial maps into a composition of elementary simplicial maps, and they construct another sequence where all maps are inclusions. They prove that the persistence diagram of (1) can be derived from the persistent diagram of the new sequence.

We tackle this problem with a different point of view by using Dowker complexes. Consider a relation $R \subset X \times Y$, and define two abstract simplicial complexes called Dowker complexes [2]. A simplex $[x_1, x_2, \dots, x_n] \in K_R$ if and only if there exists $y \in Y$ such that $(x_i, y) \in R$ for all $i = 1, 2, \dots, n$. Similarly, a simplex $[y_1, y_2, \dots, y_n] \in L_R$ if and only if there exists $x \in X$ such that $(x, y_i) \in R$ for all $i = 1, 2, \dots, n$. By the Dowker's Theorem, $|K_R|$ and $|L_R|$ are homotopically equivalent.

First, we show that we can adapt the method of elementary simplicial maps to Dowker complexes. Second, we use the Dowker's Theorem to transform (1) to a new sequence. Let $(f, g) : R \to R'$ be a pair of maps such that if $(x, y) \in R$ then $(f(x), g(y)) \in R'$. The pair (f, g) induces two simplicial maps $K_g : K_R \to K_{R'}$, and $L_f : L_R \to L_{R'}$. We can decompose $(f, g) = (f, id) \circ (id, g)$ up to some relation $\bar{R}$. For a simplicial map $f : K_1 \to K_2$, we can define the relations R_1 and R_2 such that $K_1 = K_{R_1}$, $K_2 = K_{R_2}$, and $(f, f) : R_1 \to R_2$ is well defined. We obtain a new sequence

$$K_1 = K_{R_1} \hookrightarrow K_{\bar{R}} \xrightarrow{\varphi_1} L_{\bar{R}} \hookrightarrow L_{\bar{R}_2} \xrightarrow{\varphi_2} K_{R_2} = K_2 \quad (2)$$

where φ_i are the homotopy equivalence from the Dowker's Theorem. Similar result as in [1] can be obtained, namely, one can derive the persistent diagram of (1) from the persistent diagram of (2).

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References

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