Global bifurcation results for a delay differential system representing a chemostat model

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In this talk, we present a global bifurcation result for periodic solutions of the following delayed first order system, depending on a real parameter $\lambda \geq 0$,

$$\begin{cases} s'(t) = Ds^0(t) - Ds(t) - \dfrac{\lambda}{\gamma}\,\mu(s(t))x(t) & t \geq 0 \\ x'(t) = x(t)\big[\lambda\mu(s(t-\tau)) - D\big] & t \geq 0, \end{cases} \tag{1}$$

in which the following conditions hold:

(a) $s^0 : \mathbb{R} \to \mathbb{R}$ is continuous, positive and ω-periodic, where $\omega > 0$ is given,

(b) $\mu : [0, +\infty) \to [0, +\infty)$ is C^2 and verifies $\mu(0) = 0$ and $\mu'(s) > 0$, for any $s \in [0, +\infty)$,

(c) D, γ and the delay τ are positive constants,

System (1) has been studied in [1] and it represents a chemostat model, with a delay. The chemostat is a continuous bioreactor with a constant volume, in which one or more microbial species are cultivated in a liquid medium containing a set of resources with, in particular, a specific nutrient. The maps $s(t)$ and $x(t)$ are, respectively, the densities of the nutrient and of the microbial species at time t. The device receives continuously an input of liquid volume, described by $s^0(t)$, containing a variable concentration of the specific nutrient. It expulses continuously towards the exterior an output of liquid volume containing a mixing of microbial biomass and nutrient. The model described by the system (1) assumes that the consumption of the nutrient has no immediate effects on the microbial growth, but we have a time interval $[t - \tau, t]$ in which the microbial species metabolize(s) the nutrient.

If (s, x) is any solution of (1) such that x vanishes at some t_0, then x turns out to be identically zero. Thus, the first equation in system (1) becomes linear and has a unique ω-periodic solution, which is positive and can be written as

$$v^*(t) = \int_{-\infty}^{t} e^{-D(t-r)} D s^0(r)\, dr.$$

For a sake of simplicity, assume that $\frac{1}{\omega} \int_0^\omega \mu(v^*(t))\, dt = D$.

In [1], the authors prove that

(a) *if $\lambda < 1$ (resp. $\lambda > 1$) and (s, x) is an ω-periodic solution, different from $(v^*, 0)$, then, $x(t) < 0$ (resp. $x(t) > 0$) for all $t \in \mathbb{R}$;*

(b) *if $\lambda = 1$, no ω-periodic solution is different from $(v^*, 0)$.*

Hence, it is quite natural to ask if $(v^*, 0)$ is a bifurcation point for ω-periodic solutions of (1) as well as to investigate the global behaviour of the bifurcating branches of such solutions. Here, we call *ω-triple* an element (λ, s, x) in which (s, x) is an ω-periodic solution of (1) corresponding to λ. Denote by E the Banach space $E := \mathbb{R} \times C^1_\omega \times C^1_\omega$, where

$$C^1_\omega = \{u \in C^1([0, \omega], \mathbb{R}) : u(0) = u(\omega) \text{ and } u'(0) = u'(\omega)\}.$$

Our main result is the following:

There exist in E exactly two connected components $\mathcal{C}_+$ and $\mathcal{C}_-$ of nontrivial ω-triples, which are unbounded, contain $(1, v^*, 0)$ in their closure and are such that every $(\lambda, s, x) \in \mathcal{C}_+$ verifies $\lambda > 1$, $0 < s < v^*$ and $x > 0$, while every $(\lambda, s, x) \in \mathcal{C}_-$ verifies $\lambda < 1$, $s > v^*$ and $x < 0$.

The proof uses, among other tools, the Crandall-Rabinowitz local bifurcation theorem [3] and a concept of degree introduced in [2] for Fredholm maps of index zero between Banach spaces.

References

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